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Vortex images and q-elementary functions

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Abstract
In the present paper, the problem of vortex images in the annular domain between two coaxial cylinders is solved by the q-elementary functions. We show that all images are determined completely as poles of the q-logarithmic function, and are located at sites of the q-lattice, where a dimensionless parameter $q = r_2^2/r_1^2$ is given by the square ratio of the cylinder radii. The resulting solution for the complex potential is represented in terms of the Jackson q-exponential function. Our approach in this paper provides an efficient path to rediscover known solutions for the vortex–cylinder pair problem and yields new solutions as well. By composing pairs of q-exponents to the first Jacobi theta function and conformal mapping to a rectangular domain we show that our solution coincides with the known one, obtained before by elliptic functions. The Schottky–Klein prime function for the annular domain is factorized explicitly in terms of q-exponents. The Hamiltonian, the Kirchhoff–Routh and the Green functions are constructed. As a new application of our approach, the uniformly rotating exact N-vortex polygon solutions with the rotation frequency expressed in terms of q-logarithms at Nth roots of unity are found. In particular, we show that a single vortex orbits the cylinders with constant angular velocity, given as the q-harmonic series. Vortex images in two particular geometries with only one cylinder as the $q \to \infty$ limit are studied in detail.

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1. Introduction: method of images and circle theorem

The classical method of images introduced by Thomson in 1845 has become a powerful method for solving boundary value problems in electrostatics and hydrodynamics [1–4]. The method has been successfully applied to simple geometries such as spheres, cylinders and half-spaces, where explicit formulae have been given. Unfortunately, for complex body shapes the image principle becomes extremely difficult to find even an approximate solution. This is why the image problem for which a solution can be found in an exact form, like merging cylinders [5] for example, becomes a member of a very exceptional family.

Greenhill [3] was probably the first to study the motion of one and two vortices inside and outside of the circular domain using this method. Due to the Riemann mapping theorem this allows one by conformal mapping to solve the problem for any simply connected domain. An extension of the problem to multiply connected domains is not straightforward. For a doubly connected domain, it is well known [6] that any doubly connected region can be one-to-one and conformally mapped to the annular region bounded by two circles, $r_1 < |z| < r_2$. Moreover this canonical domain is unique, up to the linear map. This means that if the domain B is mapped to two different annular domains $r_1 < |z| < r_2$ and $r_1' < |\zeta| < r_2'$, then the last ones are related by linear transformation $\zeta = \mathrm{e}^{\mathrm{i}\alpha} z$, thus have the same circles ratio $r_2/r_1 = r_2'/r_1'$ [6]. This is the reason why the problem of the vortex motion in an annular domain between two coaxial cylinders is the canonical one. Recently, it was solved by several methods. One evident approach is related with conformal mapping of the annulus to a rectangle [12, 16] and uses the image method for the last one. It produces a double periodic lattice of images, described in terms of elliptic functions [2]. Another more general approach to multiply connected domains is based on the group of Möbius transformations, and the solution is given in terms of the Schottky–Klein prime function [13–15]. The Schottky–Klein function for the annulus is related to the first Jacobi theta function and both methods give the same result. The advantage of using elliptic functions is that they are known very well and this form of the solution gives some freedom to choose representation of the theta function, which could optimize numerical and analytical computations [16]. However, this solution cannot be considered as completely satisfactory. Since the solution includes conformal mapping from another region, in this framework no clear picture of vortex images in the annulus was given. Moreover, despite the compact form of the solution, its application to the N-vortex interaction is cumbersome. From another side, due to the canonical character of the annular domain, it is interesting to find the solution reflecting the geometry of the annulus. The goal of the present paper is to find the solution of the planar vortex problem directly in the annular domain between two coaxial cylinders. We show that the two cylinders are imitated by the infinite set of vortex images located at sites of the so-called q-lattice, where $q = r_2^2/r_1^2$. Then the solution for the complex velocity and complex potential are representable naturally in terms of the q-logarithmic and q-exponential functions. These functions accurately describe four sublattices of the vortex images in two cylinders of the odd and even orders. Then combined in a proper form, they become building blocks of the Schottky–Klein prime function and the first Jacobi theta function. The image representation in our approach converges very fast, so that the number of images can be determined by the level of approximation.

The problem considered here has several interesting applications. One of them is related to hydrodynamic interaction in which the modification of ambient flow by cylinders is carried out. First, by obtaining the effect of a single cylinder on the flow, and then applying the boundary conditions to each cylinder, we can determine unknown coefficients that appear in the series expansions. The problem of water waves diffraction by multiple cylinders placed at the free surface, but without vortices, was solved in [7].

Another application is related to inviscid two-dimensional fluid dynamics experiments with magnetized electron columns confined in a cylindrical trap [24, 25]. Here the flow vorticity is proportional to the electron density and the electric potential is analogous to the two-dimensional stream function. Thus, the electrons mimic ideal two-dimensional fluid equations and by creating electron columns with the appropriate density, one can model the fluid flows. It allows one to model also the real problems of vortex interaction with topography [12]. A mathematically similar problem appears for the point vortices in liquid helium in a rotating cylinder. Or for anyons [11], the point particles interacting with the Chern–Simons field, confined in a cylindrical trap. Motion of a vortex in the neighborhood of a cosmic string [26] and influence of the cylindrically compactified extra space dimension on this vortex, is another class of possible applications in the Kaluza–Klein-type cosmological models.

1.1. The circle theorem

The one-vortex problem in the presence of a cylinder at the origin can be solved easily by the circle theorem of Milne-Thomson [1]. For complex velocity of the flow $\bar{V}(z) = u_1 - \mathrm{i}u_2$, this theorem can be reformulated in the form

$$\bar{V}(z) = \bar{v}(z) - \frac{r_1^2}{z^2} v\left(\frac{r_1^2}{z}\right), \tag{1}$$

where $v(z)$ is a complex velocity of the flow in the unbounded domain, and the second term represents correction to the complex velocity by the cylinder of radius r_1 placed at the origin. The normal velocity of the flow, proportional to $[\bar{V}(z)z + V(\bar{z})\bar{z}]$, vanishes at the surface of the cylinder, $z\bar{z} = r_1^2$. Located at position z_0, a point vortex of strength κ and corresponding circulation $\Gamma = -2\pi\kappa$, (1) can be written explicitly as

$$\bar{V}(z) = \frac{\mathrm{i}\kappa}{z - z_0} - \frac{\mathrm{i}\kappa}{z - \frac{r_1^2}{\bar{z}_0}} + \frac{\mathrm{i}\kappa}{z}, \tag{2}$$

where the second term represents a vortex of strength $-\kappa$ at the inverse point $r_1^2/\bar{z}_0$ with respect to the cylinder, and the last term is the positive strength vortex at the origin. Henceforth, the vortices at the inverse point and at the center of the cylinder, imitating the circular boundary, we shall call 'vortex images' or simply 'images'. Therefore, in (2), there are two images—one positive image at the center of the cylinder and another negative image at the inverse point. These two images are used to replace correctly the circular boundary in the infinite 2D plane.

1.2. Vortex inside a cylindrical domain

Another application is the case of a vortex at point z_0 inside a cylindrical domain with radius r_2, $\mathrm{C} : |z| < r_2$,

$$\bar{V}(z) = \frac{\mathrm{i}\kappa}{z - z_0} - \frac{\mathrm{i}\kappa}{z - \frac{r_2^2}{\bar{z}_0}}, \tag{3}$$

where the vortex image is located at point $r_2^2/\bar{z}_0$ outside C. Solution (3) can be obtained from the circle theorem by using the mapping $z = 1/\omega$.

1.3. Vortex in annular domain

The above two examples are limiting cases of the problem of a point vortex in the annular domain between two coaxial cylinders with inner radius r_1 and outer radius r_2. By the Laurent series expansion, we find in this case that the solution is given as the infinite set of images in two cylinders:

$$\bar{V}(z) = \sum_{n=-\infty}^{\infty} \left[\frac{\mathrm{i}\kappa}{z - z_0 q^n} - \frac{\mathrm{i}\kappa}{z - \frac{r_1^2}{\bar{z}_0} q^n}\right]. \tag{4}$$

As we show in this paper these images are determined completely in terms of the q-logarithmic and q-exponential functions. Mathematical study of these functions is connected with some applications in the number theory: for calculation of the Euler's constant γ by generalization of a classical formula due to Ramanujan and Vacca [19], the irrationality test [18, 19], the Stieltjes transform of a positive discrete measure [17] and construction of the Padé approximations [17]. Another class of applications is related to quantum groups and their representations [22]. Starting from the quantum integrable systems, the physical systems with such symmetries have been extended to the several q-deformed models such as the quantum linear harmonic oscillator, generalized coherent states in quantum optics, composite particles with the Chern–Simons flux—the anyons. In general, it was observed that physical systems with a fundamental length scale have a symmetry of a quantum group [8]. In nuclear physics the deformation parameter is related to the time scale of strong interactions [9], while in solid state physics with the lattice spacing [10]. Despite this particular progress, the direct interpretation of the deformation parameter in some cases is incomplete or even nonexistent. In the present paper, we study the classical problem of N vortices placed in the annular region between two coaxial cylinders with radii $r_1 < r_2$, and we find that the natural dimensionless parameter $q = r_2^2/r_1^2$, uniquely characterizing this canonical doubly connected domain, plays the role of the fundamental length, producing an infinite q-lattice of vortex images.

The paper is organized as follows. In section 2 by the Laurent series expansion we solved the N-vortex problem in the annular domain. The solution is represented in terms of the q-elementary functions which we review in section 3. In section 4 by using the q-logarithmic function, we show that all vortex images are determined by simple pole singularities of this function and form the so-called q-lattice. By using the q-exponential function in section 5, we construct a complex potential of the problem. In section 6, we find relation of our solution with previously known ones in terms of elliptic functions and the Schottky–Klein prime function. We show that both of them are factorized in terms of the q-exponential functions, known also as the quantum dilogarithm. As a first simple application of our results, we solve a single vortex problem and find its rotation frequency in terms of q-harmonic numbers (section 7). The limiting cases of the N-vortex problem in outside region and in inside region of a cylindrical domain are studied in section 8 as $q \to \infty$ limit of our solution. Section 9 is devoted to the description of N-vortex interaction in annulus. We construct the Hamiltonian structure, the Kirchhoff–Routh function and the hydrodynamic Green's function for the problem. New results based on our approach appear in section 10 for uniformly rotating regular N-vortex polygon. Then we make some deductions of our new solutions relevant in the present context as $q \to \infty$ limit. In conclusions, we briefly summarized our results and discuss some possible generalizations. In appendices A, B and C, some details of calculations are given.

2. *N* vortices in the annular domain

We consider the problem of N point vortices with strengths $\kappa_1, \ldots, \kappa_N$ at positions $z_1, \ldots, z_N$, respectively, in the annular domain $D : \{r_1 \leqslant |z| \leqslant r_2\}$. The region is bounded by two concentric circles: $C_1 : z\bar{z} = r_1^2$ and $C_2 : z\bar{z} = r_2^2$.

2.1. Laurent series and boundary conditions

The complex velocity is given by the Laurent series

$$\bar{V}(z) = \sum_{k=1}^{N} \frac{\mathrm{i}\kappa_k}{z - z_k} + \sum_{n=0}^{\infty} a_n z^n + \sum_{n=0}^{\infty} \frac{b_{n+1}}{z^{n+1}} \tag{5}$$

which has to satisfy the boundary conditions at both cylinders,

$$[z\bar{V}(z) + \bar{z}V(\bar{z})]|_{C_k} = 0, \qquad k = 1, 2. \tag{6}$$

Conditions (6) imply that no fluid can penetrate any of the circular walls of the domain. If **n** denotes the normal to the boundary, the boundary condition is that the normal velocity must be zero, $\mathbf{u}\cdot\mathbf{n} = u_n = 0$.

To find unknown coefficients we have to determine the boundary conditions for

$$\Gamma = z\bar{V}(z) + \bar{z}V(\bar{z}) = \sum_{k=1}^{N} \frac{\mathrm{i}\kappa_k z}{z - z_k} + \sum_{n=0}^{\infty} a_n z^{n+1} + \sum_{n=0}^{\infty} \frac{b_{n+2}}{z^{n+1}} + \mathrm{CC} \tag{7}$$

where CC stands for the complex conjugate.

- Since on boundary C_1: $z\bar{z} = {r_1}^2$ and $|z_k| > |z|$, equation (7) gives

$$\Gamma_{|C_1} = \sum_{k=1}^{N} (-\mathrm{i}\kappa_k) \sum_{n=0}^{\infty} \left(\frac{z}{z_k}\right)^{n+1} + \sum_{n=0}^{\infty} a_n z^{n+1} + \sum_{n=0}^{\infty} \frac{b_{n+2}}{z^{n+1}} + \mathrm{CC}$$
$$= \sum_{n=0}^{\infty} \left[\sum_{k=1}^{N} \frac{-\mathrm{i}\kappa_k}{z_k^{n+1}} + a_n + \bar{b}_{n+2} \frac{1}{r_1^{2(n+1)}} \right] z^{n+1} + \mathrm{CC} = 0. \tag{8}$$

This implies the first algebraic system of equations

$$\sum_{k=1}^{N} \frac{-\mathrm{i}\kappa_k}{z_k^{n+1}} + a_n + \bar{b}_{n+2} \frac{1}{r_1^{2(n+1)}} = 0 \quad (n = 0, 1, 2, \ldots) \qquad \text{and} \qquad b_1 = 0. \tag{9}$$

- Since on boundary C_2: $z\bar{z} = r_2^2$ and $|z_k| < |z|$, equation (7) gives

$$\Gamma_{|C_2} = \sum_{k=1}^{N} (\mathrm{i}\kappa_k) \sum_{n=0}^{\infty} \left(\frac{z_k}{z}\right)^{n} + \sum_{n=0}^{\infty} a_n z^{n+1} + \sum_{n=0}^{\infty} \frac{b_{n+2}}{z^{n+1}} + \mathrm{CC}$$
$$= \sum_{n=0}^{\infty} \left[\sum_{k=1}^{N} \frac{-\mathrm{i}\kappa_k \bar{z}_k^{n+1}}{r_2^{2(n+1)}} + a_n + \bar{b}_{n+2} \frac{1}{r_2^{2(n+1)}} \right] z^{n+1} + \mathrm{CC} = 0. \tag{10}$$

This implies the second algebraic system of equations

$$\sum_{k=1}^{N} \frac{-\mathrm{i}\kappa_k \bar{z}_k^{n+1}}{r_2^{2(n+1)}} + a_n + \bar{b}_{n+2} \frac{1}{r_2^{2(n+1)}} = 0 \quad (n = 0, 1, 2, \ldots). \tag{11}$$

2.2. *Solution of the algebraic system*

We have two algebraic systems (9) and (11). By subtracting (11) from (9), we eliminate a_n

$$\bar{b}_{n+2} \left[\frac{1}{r_2^{2(n+1)}} - \frac{1}{r_1^{2(n+1)}} \right] + \sum_{k=1}^{N} (-\mathrm{i}\kappa_k) \left[\frac{\bar{z}_k^{n+1}}{r_2^{2(n+1)}} - \frac{1}{z_k^{n+1}} \right] = 0. \tag{12}$$

If $q \equiv r_2^2 / r_1^2$ is used, we find

$$b_{n+2} = \sum_{k=1}^{N} \left(\frac{-\mathrm{i}\kappa_k}{\bar{z}_k^{n+1}} \frac{r_2^{2(n+1)} - |z_k|^{2(n+1)}}{q^{n+1} - 1} \right) \tag{13}$$

and from (9) we determine a_n,

$$a_n = \sum_{k=1}^{N} \frac{\mathrm{i}\kappa_k}{z_k^{n+1}} - \bar{b}_{n+2} \frac{1}{r_1^{2(n+1)}}, \tag{14}$$

or

$$a_n = \sum_{k=1}^{N} \frac{-\mathrm{i}\kappa_k}{z_k^{n+1}} \frac{r_1^{2(n+1)} - |z_k|^{2(n+1)}}{r_1^{2(n+1)}(q^{n+1}-1)}. \tag{15}$$

The Taylor part of (5) gives the following:

$$\begin{aligned}
\sum_{n=0}^{\infty} a_n z^n &= \sum_{n=0}^{\infty}\sum_{k=1}^{N} \frac{-\mathrm{i}\kappa_k}{z_k^{n+1}} \frac{z^n}{q^{n+1}-1} - \sum_{n=0}^{\infty}\sum_{k=1}^{N} \frac{(-\mathrm{i}\kappa_k)\bar{z}_k^{(n+1)} z^n}{r_1^{2(n+1)}(q^{n+1}-1)} \\
&= \sum_{k=1}^{N} \frac{-\mathrm{i}\kappa_k}{z(q-1)} \sum_{n=0}^{\infty} \frac{1}{[n+1]}\left(\frac{z}{z_k}\right)^{n+1} + \sum_{k=1}^{N} \frac{\mathrm{i}\kappa_k}{z(q-1)} \sum_{n=0}^{\infty} \frac{1}{[n+1]}\left(\frac{z\bar{z}_k}{r_1^2}\right)^{n+1} \\
&= \sum_{k=1}^{N} \frac{\mathrm{i}\kappa_k}{z(q-1)} \mathrm{Ln}_q\left(1-\frac{z}{z_k}\right) - \sum_{k=1}^{N} \frac{\mathrm{i}\kappa_k}{z(q-1)} \mathrm{Ln}_q\left(1-\frac{z\bar{z}_k}{r_1^2}\right)
\end{aligned} \tag{16}$$

and the Laurent part gives

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{b_{n+2}}{z^{n+2}} &= \sum_{k=1}^{N}\left(\frac{1}{z}\sum_{n=0}^{\infty} \frac{-\mathrm{i}\kappa_k}{\bar{z}_k^{n+1}} \frac{r_2^{2(n+1)}}{q^{n+1}-1}\frac{1}{z^{n+1}}\right) + \sum_{k=1}^{N}\left(\frac{1}{z}\sum_{n=0}^{\infty} \frac{\mathrm{i}\kappa_k z_k^{n+1}}{q^{n+1}-1}\frac{1}{z^{n+1}}\right) \\
&= \sum_{k=1}^{N} \frac{-\mathrm{i}\kappa_k}{z(q-1)} \sum_{n=0}^{\infty} \frac{1}{[n+1]}\left(\frac{r_2^2}{z\bar{z}_k}\right)^{n+1} + \sum_{k=1}^{N} \frac{\mathrm{i}\kappa_k}{z(q-1)} \sum_{n=0}^{\infty} \frac{1}{[n+1]}\left(\frac{z_k}{z}\right)^{n+1} \\
&= \sum_{k=1}^{N} \frac{\mathrm{i}\kappa_k}{z(q-1)} \mathrm{Ln}_q\left(1-\frac{r_2^2}{z\bar{z}_k}\right) - \sum_{k=1}^{N} \frac{\mathrm{i}\kappa_k}{z(q-1)} \mathrm{Ln}_q\left(1-\frac{z_k}{z}\right),
\end{aligned} \tag{17}$$

where $[n] \equiv (q^n-1)/(q-1)$ and $\mathrm{Ln}_q(1-x) \equiv -\sum_{n=1}^{\infty} \frac{x^n}{[n]}$, $|x| < q$, $q > 1$. In the following section, we survey some basic facts about q-elementary functions.

3. The q-logarithmic and q-exponential functions

3.1. The q-logarithmic function

By analogy with ordinary logarithmic function: $|x| \leqslant 1, x \neq -1$

$$\ln(1-x) = -\sum_{n=1}^{\infty} \frac{x^n}{n}, \tag{18}$$

the q-logarithmic function is defined as

$$\mathrm{Ln}_q(1-x) \equiv -\sum_{n=1}^{\infty} \frac{x^n}{[n]}, \qquad |x| < q, \qquad q > 1, \tag{19}$$

where for any positive integer n, the q-number is

$$[n] \equiv 1 + q + q^2 + \cdots + q^{n-1} = \frac{q^n-1}{q-1}. \tag{20}$$

In the limiting case $q \to 1$,

$$\lim_{q\to 1}[n] = n, \tag{21}$$

$$\lim_{q\to 1} \mathrm{Ln}_q(1-x) = \ln(1-x). \tag{22}$$

Our definition relates to the q-logarithmic function of Borwein [17]

$$\mathrm{L}_q(x) = \sum_{n=1}^{\infty} \frac{x^n}{[n]} \tag{23}$$

by

$$\mathrm{Ln}_q(1-x) = -\mathrm{L}_q(x) \tag{24}$$

and to that of Sondow and Zudilin [19]

$$\ln_q(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}x^n}{q^n - 1}, \qquad |x| < q, \qquad q > 1 \tag{25}$$

by

$$\mathrm{Ln}_q(1+x) = (q-1)\ln_q(1+x). \tag{26}$$

The q-derivative is defined as

$$D_q f(x) = \frac{f(qx) - f(x)}{(q-1)x}. \tag{27}$$

Taking the q-derivative of x^n

$$D_q x^n = [n]x^{n-1} \tag{28}$$

for the q-logarithmic function, we get

$$D_q\,\mathrm{Ln}_q(1-x) = -\frac{1}{1-x}. \tag{29}$$

The following representation of the q-logarithmic function [17, 19], extended to the complex domain, is crucial for our problem: let q be real, $q > 1$, then for $0 < |z| < q$ the following identity holds:

$$\mathrm{Ln}_q(1+z) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}z^n}{[n]} = (q-1)\sum_{n=1}^{\infty} \frac{z}{q^n + z}. \tag{30}$$

The proof is given by the following chain of transformations:

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{z}{q^n + z} &= z\sum_{n=1}^{\infty} \frac{1}{q^n}\frac{1}{1+zq^{-n}} = z\sum_{n=1}^{\infty}\frac{1}{q^n}\sum_{k=1}^{\infty}\frac{(-z)^{k-1}}{q^{n(k-1)}} \\
&= \sum_{k=1}^{\infty}(-1)^{k-1}z^k\sum_{n=1}^{\infty}\frac{1}{q^{nk}} = \sum_{k=1}^{\infty}(-1)^{k-1}z^k\frac{1}{q^k}\frac{1}{1-q^{-k}} = \sum_{k=1}^{\infty}\frac{(-1)^{k-1}z^k}{q^k - 1} \\
&= \frac{1}{q-1}\mathrm{Ln}_q(1+z).
\end{aligned} \tag{31}$$

3.2. *q-Exponential functions*

Next, we need Jackson's q-exponential functions defined as [17]

$$E_q(z) = \sum_{n=0}^{\infty} \frac{z^n}{[n]!}, \tag{32}$$

$$E_q^*(z) = \sum_{n=0}^{\infty} q^{n(n-1)/2}\frac{z^n}{[n]!}, \tag{33}$$

where $[n]! \equiv [1][2]\cdots[n]$ and q is real. As $q \to 1$, both functions reduce to the ordinary exponential. The application of the q-derivative gives

$$D_q E_q(x) = E_q(x), \qquad D_q E_q^*(x) = E_q^*(qx). \tag{34}$$

The function $E_q(z)$ is entire in z if $|q| > 1$, while it has a radius of convergence $|1-q|^{-1}$ if $|q| < 1$. The function $E_q^*(z)$ is entire for $|q| < 1$ and converges for $|z| < 1$ if $|q| > 1$. These two functions are related by

$$E_q(z) = E_{1/q}^*(z) \tag{35}$$

and

$$E_q(-z)E_q^*(z) = 1. \tag{36}$$

3.3. Infinite product representation

Particularly important for us is the infinite product representation [17]

$$E_q^*(z) = \prod_{k=0}^{\infty}(1 + zq^k(1-q)), \qquad |q| < 1 \tag{37}$$

and the related identity, obtained by setting $q \to 1/q$ and shifting the argument

$$\prod_{k=1}^{\infty}\left(1 - \frac{z}{q^k}\right) = \frac{1}{1-z}E_q\left(\frac{-z}{1-q^{-1}}\right), \tag{38}$$

which is entire for $|q| > 1$.

4. Vortex images and poles of q-logarithm

4.1. The q-logarithmic solution

Substituting equations (16) and (17) into (5) we get solution in the form,

$$\bar{V}(z) = \sum_{k=1}^{N} \mathrm{i}\kappa_k \frac{1}{z - z_k} + \sum_{k=1}^{N} \frac{\mathrm{i}\kappa_k}{z(q-1)}\left[\mathrm{Ln}_q\left(1 - \frac{z}{z_k}\right) - \mathrm{Ln}_q\left(1 - \frac{z\bar{z}_k}{r_1^2}\right)\right.$$
$$\left. + \mathrm{Ln}_q\left(1 - \frac{r_2^2}{z\bar{z}_k}\right) - \mathrm{Ln}_q\left(1 - \frac{z_k}{z}\right)\right]. \tag{39}$$

In terms of q-derivatives with different basis q it can be written in a more compact form:

$$\bar{V}(z) = \sum_{k=1}^{N}\left[\frac{\mathrm{i}\kappa_k}{z - z_k} - \frac{\mathrm{i}\kappa_k}{z_k}\left([\alpha]_q D_{q^\alpha}\mathrm{Ln}_q\left(1 - \frac{z}{z_k}\right) - \frac{z_k^2}{z^2}[1-\alpha]_q D_{q^{1-\alpha}}\mathrm{Ln}_q\left(1 - \frac{z_k}{z}\right)\right)\right], \tag{40}$$

where the real q-number $[\alpha]_q = (q^\alpha - 1)/(q-1), \alpha = \alpha(k) \equiv \log_q\left(|z_k|^2/r_1^2\right)$. Due to inequality $r_1 < |z_k| < 1$, parameter α is restricted by $0 < \alpha < 1$. Particular values of this parameter correspond to different positions of the vortex:

- for $\alpha = 1$ the vortex is on the outer cylinder $|z_0| = r_2$,
- for $\alpha = 0$ the vortex is on the inner cylinder $|z_0| = r_1$,
- for $\alpha = 1/2$ the vortex is at the geometric mean distance $|z_0| = \sqrt{r_1 r_2}$ (see equation (90)),

- for $\alpha = m/n$, where $m < n$ are positive integers, the vortex is at the 'generalized' mean distance $|z_0|^n = r_1^{n-m} r_2^m$.

  Expanding q-logarithm according to (30), we have

$$\bar{V}(z) = \sum_{k=1}^{N} \frac{i\kappa_k}{z - z_k} + \sum_{k=1}^{N}\sum_{n=1}^{\infty} \frac{i\kappa_k}{z - z_k q^n} - \sum_{k=1}^{N}\sum_{n=1}^{\infty} \frac{i\kappa_k}{z - q^n \frac{r_1^2}{\bar{z}_k}}$$
$$+ \sum_{k=1}^{N}\sum_{n=1}^{\infty} \left[\frac{i\kappa_k}{z} - \frac{i\kappa_k}{z - q^{-n}\frac{r_2^2}{\bar{z}_k}}\right] - \sum_{k=1}^{N}\sum_{n=1}^{\infty} \left[\frac{i\kappa_k}{z} - \frac{i\kappa_k}{z - q^{-n} z_k}\right] \tag{41}$$

or

$$\bar{V}(z) = \sum_{k=1}^{N} \frac{i\kappa_k}{z - z_k} + \sum_{k=1}^{N} \left[\sum_{n=1}^{\infty} \frac{i\kappa_k}{z - z_k q^n} + \sum_{n=1}^{\infty} \frac{i\kappa_k}{z - z_k q^{-n}}\right]$$
$$- \sum_{k=1}^{N} \left[\sum_{n=0}^{\infty} \frac{i\kappa_k}{z - \frac{r_1^2}{\bar{z}_k} q^{-n}} + \sum_{n=0}^{\infty} \frac{i\kappa_k}{z - \frac{r_2^2}{\bar{z}_k} q^{n}}\right]. \tag{42}$$

4.2. Vortex images

Equation (42) has a countable infinite number of pole singularities. These singularities can be interpreted as vortex images in two cylindrical surfaces. For simplicity let us consider a single vortex at position $z_0, r_1 < |z_0| < r_2$. Then the set of images in the cylinder C_1 we denote $z_I^{(1)}, z_I^{(2)}, \ldots$, and in the cylinder C_2 as $z_{II}^{(1)}, z_{II}^{(2)}, \ldots$, where

(i) $z_I^{(1)} = \frac{r_1^2}{\bar{z}_0}$, $\quad z_{II}^{(1)} = \frac{r_2^2}{\bar{z}_0}$,

(ii) $z_I^{(2)} = \frac{r_1^2}{\bar{z}_{II}^{(1)}} = \frac{z_0}{q}$, $\quad z_{II}^{(2)} = \frac{r_2^2}{\bar{z}_I^{(1)}} = q z_0$,

(iii) $z_I^{(3)} = \frac{r_1^2}{\bar{z}_{II}^{(2)}} = \frac{r_1^2}{\bar{z}_0}\frac{1}{q}$, $\quad z_{II}^{(3)} = \frac{r_2^2}{\bar{z}_I^{(2)}} = \frac{r_2^2}{\bar{z}_0} q$,

(iv) $z_I^{(4)} = \frac{r_1^2}{\bar{z}_{II}^{(3)}} = \frac{z_0}{q^2}$, $\quad z_{II}^{(4)} = \frac{r_2^2}{\bar{z}_I^{(3)}} = z_0 q^2$,

(v) $z_I^{(5)} = \frac{r_1^2}{\bar{z}_{II}^{(4)}} = \frac{r_1^2}{\bar{z}_0}\frac{1}{q^2}$, $\quad z_{II}^{(5)} = \frac{r_2^2}{\bar{z}_I^{(4)}} = \frac{r_2^2}{\bar{z}_0} q^2$,

(vi) ...

Combining together and taking into account alternating signs (the negative one for the first image and the positive one for the next image—the image of the image), we have two sets of consecutive images

$$z_0, \quad z_I^{-(1)}, \quad z_{II}^{+(2)}, \quad z_I^{-(3)}, \quad z_{II}^{+(4)}, \quad z_I^{-(5)}, \ldots \tag{43}$$

and

$$z_0, \quad z_{II}^{-(1)}, \quad z_I^{+(2)}, \quad z_{II}^{-(3)}, \quad z_I^{+(4)}, \quad z_{II}^{-(5)}, \ldots. \tag{44}$$

It shows that the set of vortex images is determined completely by simple pole singularities of the q-logarithmic function. By identity $r_2^2/q^n = r_1^2/q^{n-1}$, in representation (42) we can combine sums so that

$$\bar{V}(z) = \sum_{k=1}^{N} \left[\sum_{n=-\infty}^{\infty} \frac{i\kappa_k}{z - z_k q^n}\right] - \sum_{k=1}^{N} \left[\sum_{n=0}^{\infty} \frac{i\kappa_k}{z - \frac{r_1^2}{\bar{z}_k} q^{-n}} + \sum_{n=1}^{\infty} \frac{i\kappa_k}{z - \frac{r_1^2}{\bar{z}_k} q^{n}}\right] \tag{45}$$
$$= \sum_{k=1}^{N} i\kappa_k \sum_{n=-\infty}^{\infty} \left[\frac{1}{z - z_k q^n} - \frac{1}{z - \frac{r_1^2}{\bar{z}_k} q^n}\right]. \tag{46}$$

4.3. The complex potential and stream function

According to the relation

$$\bar{V}(z) = F'(z), \tag{47}$$

the complex potential of the flow we find in the form

$$F(z) = \sum_{k=1}^{N} \mathrm{i}\kappa_k \sum_{n=-\infty}^{\infty} \left[\ln(z - z_k q^n) - \ln\left(z - \frac{r_1^2}{\bar{z}_k} q^n\right)\right], \tag{48}$$

or compactly

$$F(z) = \sum_{k=1}^{N} \mathrm{i}\kappa_k \sum_{n=-\infty}^{\infty} \left[\ln \frac{z - z_k q^n}{z - \frac{r_1^2}{\bar{z}_k} q^n}\right]. \tag{49}$$

This gives a clear picture of the structure of vortex images. For a given vortex at z_k, the first positive sum in (48) represents the contribution from the vortex and the infinite number of its even images, while the second, negative sum corresponds to odd images. The infinite set of points

$$\ldots, q^{-n} z_k, \ldots, q^{-2} z_k, \qquad q^{-1} z_k, z_k, q z_k, \qquad q^2 z_k, \ldots, q^n z_k, \ldots \tag{50}$$

we call the q-chain. Then the set of vortex images in (48) forms the vortex q-lattice, consisting of two q-chains generated by vortex κ_k at z_k and its image $-\kappa_k$ at $r_1^2/\bar{z}_0$.

The stream function $\psi = (F - \bar{F})/2\mathrm{i}$ in terms of vortex images is

$$\psi(z) = \sum_{k=1}^{N} \kappa_k \sum_{n=-\infty}^{\infty} \ln \left|\frac{z - z_k q^n}{z - \frac{r_1^2}{\bar{z}_k} q^n}\right|. \tag{51}$$

However, formal consideration of the infinite chain of vortices frequently leads to divergency. For convergency of the problem one needs to add a constant term and regroup terms. In our case to rewrite the result in convergent form, we can use the Jackson q-exponential function (32).

5. The complex potential and q-exponential function

Now we use a new function defined in [17] as

$$f_q(z) \equiv \prod_{n=1}^{\infty} \left(1 - \frac{z}{q^n}\right) = \frac{E_q\left(\frac{zq}{1-q}\right)}{1 - z}, \tag{52}$$

where $|q| > 1$ and observe that

$$\frac{f_q'(z)}{f_q(z)} = \frac{\mathrm{d}}{\mathrm{d}z} \ln f_q(z) = -\sum_{n=1}^{\infty} \frac{q^{-n}}{1 - zq^{-n}} = \sum_{n=1}^{\infty} \frac{1}{z - q^n}. \tag{53}$$

From (30) we get its relation with the q-logarithmic function:

$$\frac{\mathrm{Ln}_q(1-z)}{(q-1)z} = \frac{\mathrm{d}}{\mathrm{d}z} \ln f_q(z) = \frac{\mathrm{d}}{\mathrm{d}z} \ln \frac{E_q\left(\frac{qz}{1-q}\right)}{1 - z}. \tag{54}$$

By using the q-derivative of the q-exponential function (34) and rescaling the argument

$$E_q\left(\frac{qz}{1-q}\right) = (1 - z) E_q\left(\frac{z}{1-q}\right), \tag{55}$$

this expression can be simplified so that

$$\frac{\mathrm{Ln}_q(1-\alpha z)}{(q-1)z} = \frac{\mathrm{d}}{\mathrm{d}z} \ln E_q\left(\frac{\alpha z}{1-q}\right). \tag{56}$$

By similar arguments we also find

$$\frac{\mathrm{Ln}_q\left(1-\frac{\alpha}{z}\right)}{(q-1)z} = -\frac{\mathrm{d}}{\mathrm{d}z} \ln E_q\left(\frac{\alpha}{(1-q)z}\right). \tag{57}$$

Applying these formulae to (39) we get for complex velocity

$$\bar{V}(z) = \sum_{k=1}^{N} \mathrm{i}\kappa_k \frac{\mathrm{d}}{\mathrm{d}z} \ln\left[(z-z_k)\frac{E_q\left(\frac{z}{(1-q)z_k}\right)E_q\left(\frac{z_k}{(1-q)z}\right)}{E_q\left(\frac{z\bar{z}_k}{(1-q)r_1^2}\right)E_q\left(\frac{r_2^2}{(1-q)z\bar{z}_k}\right)}\right], \tag{58}$$

which implies complex potential in the form

$$F(z) = \sum_{k=1}^{N} \mathrm{i}\kappa_k \left[\ln(z-z_k) + \ln\frac{E_q\left(\frac{z}{(1-q)z_k}\right)E_q\left(\frac{z_k}{(1-q)z}\right)}{E_q\left(\frac{z\bar{z}_k}{(1-q)r_1^2}\right)E_q\left(\frac{r_2^2}{(1-q)z\bar{z}_k}\right)}\right]. \tag{59}$$

The first term in the bracket corresponds to the vortex at position z_k while the second term describes its images. As we can see these images are completely determined by zeros of the q-exponential functions. For the zeros of the q-exponential function and asymptotic formulae for varying parameter q, see [21].

6. Conformal mapping and theta function

In this section, we compare our solution (59) with that of Johnson and McDonald [12] and Crowdy and Marshall [14]. For comparison purposes we fix the radius $r_2 = 1$ so that $q = r_2^2/r_1^2 = 1/r_1^2 \equiv 1/\tilde{q}^2$, and new parameter $\tilde{q} < 1$. Then according to (35), $E_q(z) = E^*_{\tilde{q}^2}(z)$, so that the complex potential (59) is representable in terms of the second Jackson q-exponentials as

$$F(z) = \sum_{k=1}^{N} \mathrm{i}\kappa_k \ln\left[(z-z_k)\frac{E^*_{\tilde{q}^2}\left(\frac{\tilde{q}^2 z}{(\tilde{q}^2-1)z_k}\right)E^*_{\tilde{q}^2}\left(\frac{\tilde{q}^2 z_k}{(\tilde{q}^2-1)z}\right)}{E^*_{\tilde{q}^2}\left(\frac{z\bar{z}_k}{(\tilde{q}^2-1)}\right)E^*_{\tilde{q}^2}\left(\frac{\tilde{q}^2}{(\tilde{q}^2-1)z\bar{z}_k}\right)}\right]. \tag{60}$$

By (37), q-exponential functions can be represented as infinite products

$$E^*_{\tilde{q}^2}\left(\frac{z}{\tilde{q}^2-1}\right) = \prod_{n=0}^{\infty}(1-\tilde{q}^{2n}z) \equiv (1-z)^{\infty}_{\tilde{q}^2}, \tag{61}$$

$$E^*_{\tilde{q}^2}\left(\frac{\tilde{q}^2 z}{\tilde{q}^2-1}\right) = \prod_{n=1}^{\infty}(1-\tilde{q}^{2n}z) = \frac{(1-z)^{\infty}_{\tilde{q}^2}}{1-z}. \tag{62}$$

6.1. Jacobi theta function

The first Jacobi theta function is defined as an infinite product [23]

$$\Theta_1(x;\tilde{q}) = 2G_{\tilde{q}}\tilde{q}^{\frac{1}{4}} \sin x \prod_{n=1}^{\infty}(1-\tilde{q}^{2n}\,\mathrm{e}^{2\mathrm{i}x})(1-\tilde{q}^{2n}\,\mathrm{e}^{-2\mathrm{i}x}), \tag{63}$$

where

$$G_{\tilde{q}} \equiv \prod_{n=1}^{\infty}(1-\tilde{q}^{2n}) \tag{64}$$

$(\tilde{q} < 1)$ or

$$\Theta_1(x;\tilde{q}) = \frac{G\tilde{q}^{\frac{1}{4}}}{2\sin x}(1-\mathrm{e}^{2\mathrm{i}x})_{\tilde{q}^2}^{\infty}(1-\mathrm{e}^{-2\mathrm{i}x})_{\tilde{q}^2}^{\infty}. \tag{65}$$

According to (61) and (62), this theta function can be factorized to two q-exponentials and an elementary function

$$\Theta_1(x;\tilde{q}) = 2G\tilde{q}^{\frac{1}{4}}\sin x\, E_{\tilde{q}^2}^{*}\left(\frac{\tilde{q}^2\mathrm{e}^{2\mathrm{i}x}}{\tilde{q}^2-1}\right)E_{\tilde{q}^2}^{*}\left(\frac{\tilde{q}^2\mathrm{e}^{-2\mathrm{i}x}}{\tilde{q}^2-1}\right). \tag{66}$$

Then the complex potential becomes

$$F(z) = \sum_{k=1}^{N}\mathrm{i}\kappa_k \ln\left[z\frac{\left(1-\frac{1}{z\bar{z}_k}\right)}{\left(1-\frac{z}{z_k}\right)}\frac{\left(1-\frac{z}{z_k}\right)_{\tilde{q}^2}^{\infty}\left(1-\frac{z_k}{z}\right)_{\tilde{q}^2}^{\infty}}{(1-z\bar{z}_k)_{\tilde{q}^2}^{\infty}\left(1-\frac{1}{z\bar{z}_k}\right)_{\tilde{q}^2}^{\infty}}\right]. \tag{67}$$

Denoting

$$\frac{z}{z_k} = \mathrm{e}^{2\mathrm{i}u_k}, \qquad z\bar{z}_k = \mathrm{e}^{2\mathrm{i}v_k}, \qquad z_k\bar{z}_k = \frac{\mathrm{e}^{2\mathrm{i}v_k}}{\mathrm{e}^{2\mathrm{i}u_k}}, \tag{68}$$

where $k = 1, \ldots, N$, we obtain

$$F(z) = \sum_{k=1}^{N}\mathrm{i}\kappa_k \ln\left[z_k\,\mathrm{e}^{2\mathrm{i}u_k}\frac{(1-\mathrm{e}^{-2\mathrm{i}v_k})}{(1-\mathrm{e}^{2\mathrm{i}u_k})}\frac{(1-\mathrm{e}^{2\mathrm{i}u_k})_{\tilde{q}^2}^{\infty}(1-\mathrm{e}^{-2\mathrm{i}u_k})_{\tilde{q}^2}^{\infty}}{(1-\mathrm{e}^{2\mathrm{i}v_k})_{\tilde{q}^2}^{\infty}(1-\mathrm{e}^{-2\mathrm{i}v_k})_{\tilde{q}^2}^{\infty}}\right]. \tag{69}$$

Then by using (65) we have

$$F(z) = \sum_{k=1}^{N}\mathrm{i}\kappa_k\left[\ln\left[\frac{\Theta_1(u_k,\tilde{q})}{\Theta_1(v_k,\tilde{q})}\right]+\ln\left[-\left(\frac{z_k}{\bar{z}_k}\right)^{1/2}\right]\right]. \tag{70}$$

6.2. Conformal mapping to a rectangle

In terms of the new coordinates $\tau \equiv -\ln z$, $\tau_k \equiv -\ln z_k$ and $\bar{\tau}_k \equiv -\ln\bar{z}_k$ (conformally mapping the annulus in the z plane to a rectangle in the τ plane), we find

$$F(z) = \sum_{k=1}^{N}\mathrm{i}\kappa_k\ln\left[\frac{\Theta_1\left(\mathrm{i}\frac{\tau-\tau_k}{2},\tilde{q}\right)}{\Theta_1\left(\mathrm{i}\frac{\tau+\bar{\tau}_k}{2},\tilde{q}\right)}\right]+F_0, \tag{71}$$

where F_0 is a real constant

$$F_0 = -\frac{\mathrm{i}}{2}\sum_{k=1}^{N}\kappa_k(\tau_k-\bar{\tau}_k)-\pi\sum_{k=1}^{N}\kappa_k \tag{72}$$

and the branch for logarithm is chosen such that $\ln(-1) = \mathrm{i}\pi$. Finally, for the stream function we have

$$\Psi = \frac{F-\bar{F}}{2\mathrm{i}} = \sum_{k=1}^{N}\kappa_k\ln\left|\frac{\Theta_1\left(\frac{\mathrm{i}}{2}(\tau-\tau_k),\tilde{q}\right)}{\Theta_1\left(\frac{\mathrm{i}}{2}(\tau+\bar{\tau}_k),\tilde{q}\right)}\right|. \tag{73}$$

This coincides precisely with the result of Johnson and McDonald [12], equation (2.11). Here we would like to stress that though our solution is equivalent to the one obtained before, it has several peculiarities. Factorization (66) of the first Jacobi theta function in terms of the pair of

q-exponential functions show that the last ones are more elementary objects. They correspond to two sublattices of the vortex image lattice and seem more adequate to the geometry of the problem. According to our solution, the q-lattice of all vortex images (including vortex itself) characterizing the canonical doubly connected region, splits on four sublattices. Two sublattices are determined by even images (including vortex itself) having positive vortex strength in two cylinders, while another two sublattices correspond to odd images in both cylinders.

Practically, it is easier to manipulate with q-functions than with elliptic functions. We demonstrate this in sections 8 and 11 by considering limiting reductions to the one-cylinder problem, appearing as $q \to \infty$ limit of the annular domain. And in sections 7 and 10, we find the explicit form for the angular velocity of an N-vertex polygon in terms of the q-logarithm function. Another advantage of our form of the solution is that due to the direct image interpretation in the annulus, consideration of an approximate solution by restricting the number of images, gives the multipole expansion in vortex images. This approximation converges very fast and is useful for numerical calculations. The image structure of our solution can be applied to the similar Dirichlet problem but in arbitrary n dimensions, for the spherical shell and it cannot be seen from the elliptic function solution. Finally, involving the q-calculus to study the classical vortex problem, can be fruitful for both of them.

6.3. The Schottky–Klein prime function

To establish a link between our solution and the one by Crowdy and Marshall [14], we note that for the annular domain $\tilde{q} < |\zeta| < 1$ the Schottky group is generated by the Möbius map [14]

$$\theta_1(\zeta) = \tilde{q}^2\zeta \tag{74}$$

and its inverse. Then the Schottky–Klein prime function is

$$\omega(\zeta,\gamma) = -\frac{\gamma}{G_{\tilde{q}}^2} P(\zeta/\gamma, \tilde{q}), \tag{75}$$

where

$$P(\zeta,\tilde{q}) \equiv (1-\zeta)\prod_{n=1}^{\infty}(1-\tilde{q}^{2n}\zeta)(1-\tilde{q}^{2n}\zeta^{-1}) \tag{76}$$

and the constant $G_{\tilde{q}}$ is defined in (64). By using (62), we can factorize the Schottky–Klein prime function in terms of the Jackson q-exponential functions (33)

$$P(\zeta,\tilde{q}) = (1-\zeta)E^*_{\tilde{q}^2}\left(\frac{\tilde{q}^2}{\tilde{q}^2-1}\zeta\right)E^*_{\tilde{q}^2}\left(\frac{\tilde{q}^2}{\tilde{q}^2-1}\frac{1}{\zeta}\right). \tag{77}$$

Substituting to the complex potential (60)

$$F(z) = \sum_{k=1}^{N} \mathrm{i}\kappa_k \ln \frac{-z_k\left(1-\frac{z}{z_k}\right)E^*_{\tilde{q}^2}\left(\frac{\tilde{q}^2}{\tilde{q}^2-1}\frac{z}{z_k}\right)E^*_{\tilde{q}^2}\left(\frac{\tilde{q}^2}{\tilde{q}^2-1}\frac{z_k}{z}\right)}{(1-z\bar{z}_k)E^*_{\tilde{q}^2}\left(\frac{\tilde{q}^2}{\tilde{q}^2-1}z\bar{z}_k\right)E^*_{\tilde{q}^2}\left(\frac{\tilde{q}^2}{\tilde{q}^2-1}\frac{1}{z\bar{z}_k}\right)}, \tag{78}$$

we represent it as

$$F(z) = \sum_{k=1}^{N} \mathrm{i}\kappa_k \ln \frac{-z_k P\left(\frac{z}{z_k},\tilde{q}\right)}{P(z\bar{z}_k,\tilde{q})}. \tag{79}$$

The corresponding stream function

$$\psi = \sum_{k=1}^{N} \kappa_k \ln\left|\frac{z_k P\left(\frac{z}{z_k},\tilde{q}\right)}{P(z\bar{z}_k,\tilde{q})}\right| \tag{80}$$

is precisely the one given by Crowdy and Marshall [14], equation (8.20). Our construction of the Schottky–Klein prime function (77) suggests that this function is a composed object of more elementary q-exponential functions, which could have importance in the theory of automorphic functions and for the multiply connected domains. In fact after completing this paper Zudilin informed us that the q-exponential function, which plays a crucial role in our solution, is also known as the quantum dilogarithm [35]. It turns out that this quantum analog of the dilogarithm function is deeply involved in the quantum theory of the Teichmüller space of punctured surfaces, where punctures are similar to vortices on the Riemann surfaces.

7. The motion of a point vortex in the annular domain

7.1. Equation of motion

As an application of the above formulae we consider the motion of a single vortex in the annular domain. Complex velocity at the vortex position is determined by

$$\dot{z}_0 = \dot{x}_0 + \mathrm{i}\dot{y}_0 = V_0(\bar{z})|_{z=z_0}, \tag{81}$$

where in

$$\begin{aligned}\bar{V}_0(z) &= \frac{\mathrm{i}\kappa}{z(q-1)}\left[\mathrm{Ln}_q\left(1-\frac{z}{z_0}\right) - \mathrm{Ln}_q\left(1-\frac{z_0}{z}\right) + \mathrm{Ln}_q\left(1-\frac{r_2^2}{z\bar{z}_0}\right) - \mathrm{Ln}_q\left(1-\frac{z\bar{z}_0}{r_1^2}\right)\right]\\ &= \sum_{n=\pm 1}^{\pm\infty}\frac{\mathrm{i}\kappa}{z - z_0 q^n} - \sum_{n=\pm 1}^{\pm\infty}\frac{\mathrm{i}\kappa}{z - \frac{r_1^2}{\bar{z}_0}q^n}\end{aligned} \tag{82}$$

contribution of the vortex on itself is excluded. Taking into account that q-harmonic series [20]

$$H(q) \equiv \sum_{n=1}^{\infty}\frac{1}{[n]} = -\mathrm{Ln}_q 0 \tag{83}$$

converges for $q > 1$ and at $z = z_0$ the first two terms cancel each other, we get equations of motion

$$\dot{z}_0 = \frac{\mathrm{i}\kappa}{\bar{z}_0(q-1)}\left[\mathrm{Ln}_q\left(1-\frac{|z_0|^2}{r_1^2}\right) - \mathrm{Ln}_q\left(1-\frac{r_2^2}{|z_0|^2}\right)\right]. \tag{84}$$

7.2. Uniform rotation

From the last equation we conclude

$$\bar{z}_0\dot{z}_0 + z_0\dot{\bar{z}}_0 = \frac{\mathrm{d}}{\mathrm{d}t}|z_0|^2 = 0 \to |z_0| = \text{const}. \tag{85}$$

It implies that the distance of the vortex from the origin is a constant of motion. Substituting $z_0(t) = |z_0|\mathrm{e}^{\mathrm{i}\varphi(t)}$ into (84), we get

$$\varphi(t) = \omega t + \varphi_0, \tag{86}$$

where the frequency ω is a constant, depending on the modulus $|z_0|$,

$$\omega = \frac{\kappa}{|z_0|^2(q-1)}\left[\mathrm{Ln}_q\left(1-\frac{|z_0|^2}{r_1^2}\right) - \mathrm{Ln}_q\left(1-\frac{r_2^2}{|z_0|^2}\right)\right]. \tag{87}$$

So we find that the vortex uniformly rotates around the origin,

$$z_0(t) = |z_0|\,\mathrm{e}^{\mathrm{i}\omega t + \mathrm{i}\varphi_0} = z_0(0)\,\mathrm{e}^{\mathrm{i}\omega t}, \tag{88}$$

with rotation frequency (87) depending on the strength and initial position of the vortex and the geometry of the annular domain. This motion results from the interaction of the vortex with an infinite set of its images in the cylinders, acting as the centrifugal force in the radial direction.

7.3. Stationary vortex and geometric mean distance

Frequency (87) vanishes when

$$\mathrm{Ln}_q\left(1-\frac{|z_0|^2}{r_1^2}\right)=\mathrm{Ln}_q\left(1-\frac{r_2^2}{|z_0|^2}\right) \tag{89}$$

or $|z_0|^4 = r_1^2 r_2^2$. Then at the geometric mean distance

$$|z_0|=\sqrt{r_1 r_2} \tag{90}$$

vortex is at rest. The equation has a simple geometrical interpretation: given any two intersection points of cylinders with a ray (say $r_1 \mathrm{e}^{\mathrm{i}\alpha}$ and $r_2 \mathrm{e}^{\mathrm{i}\alpha}$) are images of each other in a cylinder with radius $\sqrt{r_1 r_2}$. Moreover, any pair of their images $r_1 \mathrm{e}^{\mathrm{i}\alpha} q^{n/2}$ and $r_2 \mathrm{e}^{\mathrm{i}\alpha} q^{-n/2}$, $n = 1, 2, \ldots,$ are also images of each other in the same cylinder. At the mean distance, angular velocity changes the sign and for the vortex approaching cylinders, ω grows as

- $\omega \to \kappa\big/\left(2r_1^2\epsilon\right)$ when $|z_0| \to r_1\,\mathrm{e}^{\epsilon} \approx r_1(1+\epsilon), 0 < \epsilon \ll 1$,
- $\omega \to -\kappa\big/\left(2r_2^2\epsilon\right)$ when $|z_0| \to r_2\,\mathrm{e}^{-\epsilon} \approx r_2(1-\epsilon), 0 < \epsilon \ll 1$.

7.4. Frequency irrationality and q-harmonic series

Here we like to indicate an intriguing relation of our solution with the number theory. Frequency (87) is an addition of two opposite sign frequencies, $\omega = \omega_1 + \omega_2$, expressed in terms of the q-logarithm functions

$$\omega_1=\frac{\kappa}{|z_0|^2(q-1)}\mathrm{Ln}_q\left(1-\frac{|z_0|^2}{r_1^2}\right)=\frac{-\kappa}{|z_0|^2(q-1)}\sum_{n=1}^{\infty}\frac{\left(\frac{|z_0|}{r_1}\right)^{2n}}{[n]}=\sum_{n=1}^{\infty}\omega_1^{(n)}, \tag{91}$$

$$\omega_2=-\frac{\kappa}{|z_0|^2(q-1)}\mathrm{Ln}_q\left(1-\frac{r_2^2}{|z_0|^2}\right)=\frac{\kappa}{|z_0|^2(q-1)}\sum_{n=1}^{\infty}\frac{\left(\frac{r_2}{|z_0|}\right)^{2n}}{[n]}=\sum_{n=1}^{\infty}\omega_2^{(n)}. \tag{92}$$

In this representation every frequency is an infinite superposition of partial frequencies coming from every vortex image. Moreover,

- for $|z_0| = r_1$ the frequency $\omega_1 = H(q)$,
- for $|z_0| = r_2$ the frequency $\omega_2 = -H(q)$

(for simplicity coefficients are taken to be unity), where $H(q)$ is the q-harmonic series. Then contribution of the first N images to the frequency ω (for N image approximation, see (148)) are given by q-harmonic numbers [27]

$$\omega^{(N)}=H_N(q)=\sum_{n=1}^{N}\frac{1}{[n]_q}. \tag{93}$$

Due to (89), at the geometric mean distance (90), frequencies (91) and (92) are compensating each other. In the annular regions

- $r_1 < |z_0| < \sqrt{r_1 r_2}, \omega_1 > |\omega_2|$ and $\omega > 0$,
- $\sqrt{r_1 r_2} < |z_0| < r_2, \omega_1 < |\omega_2|$ and $\omega < 0$.

If we fix the geometry by the value of parameter $q \geqslant 2$ and consider the unit strength vortex at a distance $|z_0|$, commensurable with one of the radii r_1 or r_2, so that the argument of the logarithm with $r = |z_0|^2\big/r_1^2$ in (91), or $r = r_2^2\big/|z_0|^2$ in (92) is a nonzero rational, then the problem of the frequency rationality is related to the problem of q-logarithm rationality. Starting from the early result of Erdos for $q = 2$ it was proved that for $q \geqslant 2$ the $\mathrm{Ln}_q(1-r)$ is irrational [18]. We could expect a relation between this irrationality and a character of the vortex dynamics similar to the one in the chaotic systems.

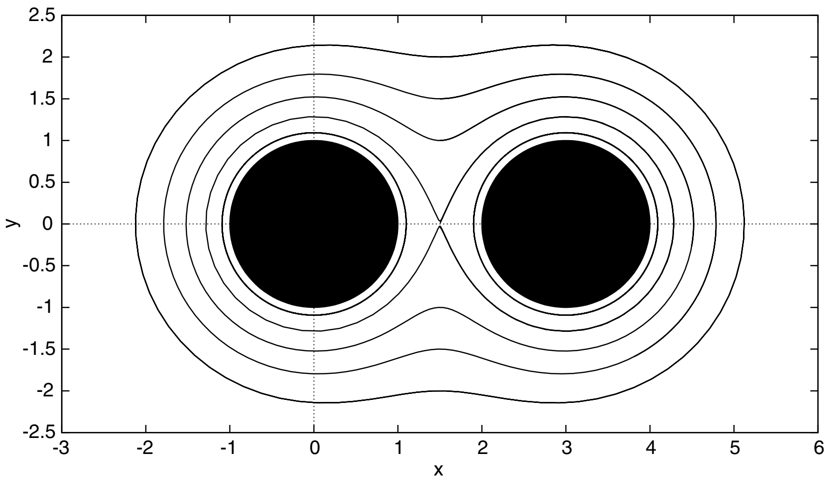

Figure 1. Trajectory of vortex motion around two cylinders.

7.5. Vortex motion outside two cylinders

The problem of a point vortex motion in a region external to two cylinders can be solved by conformal mapping of the region to the annular domain. By the Möbius transformation

$$t(z) = w = \frac{z-a}{az-1}, \tag{94}$$

we can find the complex velocity $\bar{V}(z)$ of the problem. In this mapping, the outer circle of unit radius is mapped onto itself and the inner circle of radius R_0 is mapped to a circle with radius ρ whose center is at c, $c > 0$. The real number a appearing in (96) is a geometrical parameter determined by

$$a = \frac{1 + c^2 - \rho^2 + \sqrt{((c+\rho)^2 - 1)((c-\rho)^2) - 1}}{2}. \tag{95}$$

Formulae (94) and (95) are given by [34]. As an illustration, in figure 1 we plotted trajectories of motion of one vortex around two equal cylinders of unit radius.

8. The one-vortex problem in the $q \to \infty$ limiting cases

In q-calculus the limit $q \to 1$ corresponds to a reduction of q-elementary functions to the standard elementary functions. For our problem this limit is not interesting since $r_1 = r_2$ and the annular region reduces to the circle. However, for us more interesting is another limit when $q \to \infty$. To study this problem, we need to find corresponding limits of q-elementary functions.

8.1. Asymptotic for the q-logarithm

Expanding the q-logarithm for large q, we get

$$\mathrm{Ln}_q(1+z) = (q-1)\sum_{n=1}^{\infty}\frac{z}{q^n+z} = \left(1-\frac{1}{q}\right)\sum_{n=1}^{\infty}\frac{z}{q^{n-1}}\left(1-\frac{z}{q^n}+\cdots\right). \tag{96}$$

This implies

$$\lim_{q\to\infty} \mathrm{Ln}_q(1+z) = z. \tag{97}$$

Using the definition of the q-derivative and (29),

$$\frac{\mathrm{Ln}_q(1-qz) - \mathrm{Ln}_q(1-z)}{(q-1)z} = D_q \,\mathrm{Ln}_q(1-z) = -\frac{1}{1-z}, \tag{98}$$

by (97) we get another limit

$$\lim_{q\to\infty} \frac{\mathrm{Ln}_q(1-qz)}{q-1} = \frac{z}{z-1}. \tag{99}$$

8.2. *Asymptotic behavior of the q-exponential function*

When q is large, we have

$$[n] = \frac{q^n - 1}{q-1} \approx q^{n-1} \tag{100}$$

$$[n]! = [1]\cdot[2]\cdot[3]\cdots[n] \approx q\cdot q^2\cdot q^3\cdots q^{n-1} = q^{n(n-1)/2} \tag{101}$$

and

$$E_q(z) = \sum_{n=0}^{\infty} \frac{z^n}{[n]!} \approx \sum_{n=0}^{\infty} \frac{z^n}{q^{n(n-1)/2}} = 1 + z + \frac{z^2}{q} + \cdots, \tag{102}$$

so that

$$\lim_{q\to\infty} E_q(z) = 1 + z. \tag{103}$$

Applying the q-derivative

$$\frac{E_q(qz) - E_q(z)}{(q-1)z} = D_q E_q(z) = E_q(z), \tag{104}$$

we have identities

$$E_q\left(\frac{qz}{q-1}\right) = (1+z)E_q\left(\frac{z}{q-1}\right) \tag{105}$$

$$E_q\left(\frac{z}{q}\right) = \frac{E_q(z)}{1 + \frac{q-1}{q}z}. \tag{106}$$

In the limit $q \to \infty$ they imply

$$\lim_{q\to\infty} E_q\left(\frac{z}{q}\right) = 1, \tag{107}$$

$$\lim_{q\to\infty} E_q\left(\frac{qz}{q-1}\right) = 1 + z. \tag{108}$$

8.3. *Two limiting geometries*

Since $q = r_2^2/r_1^2$, for the limit $q \to \infty$ we have two different geometrical cases.

8.3.1. Single cylinder and a vortex outside. When r_1 = constant and $r_2^2 = qr_1^2 \to \infty$, the outer cylinder becomes infinitely large so that we have just the one-cylinder problem. By replacement $r_2^2 = qr_1^2$ in (39), and using (97), (99) as $q \to \infty$, we get the finite result

$$\bar{V}(z) = \frac{\mathrm{i}\kappa}{z - z_0} - \frac{\mathrm{i}\kappa}{z}\frac{r_1^2/\bar{z}_0}{z - r_1^2/\bar{z}_0} \tag{109}$$

which exactly matches the one of the circle theorem (2). Using (87) and the limits (97), (99), for the angular velocity we have

$$\omega = \frac{\kappa r_1^2}{|z_0|^2\left(|z_0|^2 - r_1^2\right)}. \tag{110}$$

Applying limits (107), (108) to the complex potential

$$F(z) = \mathrm{i}\kappa \ln\left[(z - z_0)\frac{E_q\left(\frac{z}{(1-q)z_0}\right)E_q\left(\frac{z_0}{(1-q)z}\right)}{E_q\left(\frac{z\bar{z}_0}{(1-q)r_1^2}\right)E_q\left(\frac{qr_1^2}{(1-q)z\bar{z}_0}\right)}\right], \tag{111}$$

we obtain two images as required by the circle theorem

$$F(z) = \mathrm{i}\kappa\left[\ln(z - z_0) - \ln\left(z - \frac{r_1^2}{\bar{z}_0}\right) + \ln z\right]. \tag{112}$$

8.3.2. Single cylinder and a vortex inside. When $r_2 =$ constant and $r_1^2 = r_2^2/q \to 0$, the inner cylinder decreases until the thin string and then disappears. Replacing $r_1^2 = r_2^2/q$, and using (97), (99) in this limiting case for complex velocity (39), we get expression (3). From (87) by limits (97), (99) we have for the angular velocity of one vortex

$$\omega = -\frac{\kappa}{r_2^2 - |z_0|^2}. \tag{113}$$

Our solution coincides with the known result for single vortex inside of cylinder (see [29], example 3.7). Applying limits (107) and (108) to the complex potential (111), we obtain just the one image

$$F(z) = \mathrm{i}\kappa\left[\ln(z - z_0) - \ln\left(z - \frac{r_2^2}{\bar{z}_0}\right) + \ln\left(-\frac{r_2^2}{\bar{z}_0}\right)\right] \tag{114}$$

as derived in (3). It was shown by Greenhill [3] that single vortex rotates stationary along the circle concentric with cylindrical boundary.

9. *N*-vortex dynamics in the annular domain

Now we apply our results to the problem of N point vortices with circulations $\Gamma_1, \ldots, \Gamma_N$ ($\Gamma_j = -2\pi\kappa_j$, $j = 1, \ldots, N$), in the annular region $D = \{z, r_1 < |z| < r_2\}$.

9.1. Equations of motion

To find velocity of a vortex at z_k, following general prescription, from given complex velocity of the flow $\bar{V}(z)$ it is necessary to subtract the self-interaction of the vortex

$$\dot{\bar{z}}_k = \lim_{z \to z_k}\left(\bar{V}(z) - \frac{\Gamma_k}{2\pi\mathrm{i}}\frac{1}{z - z_k}\right). \tag{115}$$

Substitution of $\bar{V}$ from (46) gives the system of equations

$$\dot{\bar{z}}_k = \frac{1}{2\pi\mathrm{i}}\sum_{j=1 (j\neq k)}^{N}\frac{\Gamma_j}{z_k - z_j} + \frac{1}{2\pi\mathrm{i}}\sum_{j=1}^{N}\sum_{n=\pm 1}^{\pm\infty}\frac{\Gamma_j}{z_k - z_j q^n} - \frac{1}{2\pi\mathrm{i}}\sum_{j=1}^{N}\sum_{n=-\infty}^{\infty}\frac{\Gamma_j}{z_k - \frac{r_1^2}{\bar{z}_k}q^n} \tag{116}$$

$(k = 1, \ldots, N)$. To avoid the convergency problem with infinite series it would be convenient to also use the complex velocity (39) in terms of q-logarithms, so that

$$\dot{\bar{z}}_k = \frac{1}{2\pi \mathrm{i}} \sum_{j=1(j\neq k)}^{N} \frac{\Gamma_j}{z_k - z_j} + \frac{1}{2\pi \mathrm{i}} \sum_{j=1}^{N} \frac{\Gamma_j}{z_k(q-1)} \left[\mathrm{Ln}_q \left(1 - \frac{z_k}{z_j}\right) - \mathrm{Ln}_q \left(1 - \frac{z_j}{z_k}\right)\right]$$
$$- \frac{1}{2\pi \mathrm{i}} \sum_{j=1}^{N} \frac{\Gamma_j}{z_k(q-1)} \left[\mathrm{Ln}_q \left(1 - \frac{z_k \bar{z}_j}{r_1^2}\right) - \mathrm{Ln}_q \left(1 - \frac{r_2^2}{z_k \bar{z}_j}\right)\right] (k = 1, \ldots, N). \tag{117}$$

9.2. The Hamiltonian structure

The above equations of motion can be written in the Hamiltonian form

$$\Gamma_k \dot{z}_k = -2\mathrm{i}\frac{\partial H}{\partial \bar{z}_k}, \qquad \Gamma_k \dot{\bar{z}}_k = 2\mathrm{i}\frac{\partial H}{\partial z_k} \quad (k = 1, \ldots, N) \tag{118}$$

or

$$\dot{z}_k = \{z_k, H\}, \qquad \dot{\bar{z}}_k = \{\bar{z}_k, H\}, \tag{119}$$

where the Poisson bracket is defined as

$$\{f, g\} = \frac{2}{\mathrm{i}} \sum_{j=1}^{N} \frac{1}{\Gamma_j} \left(\frac{\partial f}{\partial z_j}\frac{\partial g}{\partial \bar{z}_j} - \frac{\partial f}{\partial \bar{z}_j}\frac{\partial g}{\partial z_j}\right). \tag{120}$$

This leads to the canonical bracket

$$\{z_k, \bar{z}_j\} = -\frac{2\mathrm{i}}{\Gamma_k}\delta_{kj}, \tag{121}$$

where $z_k = x_k + \mathrm{i}y_k$, $\partial/\partial z_k = (\partial/\partial x_k - \mathrm{i}\partial/\partial y_k)/2$. The Hamiltonian function for equations (116) is

$$H = -\frac{1}{4\pi} \sum_{i,j=1(i\neq j)}^{N} \Gamma_i \Gamma_j \ln|z_i - z_j| - \frac{1}{4\pi} \sum_{i,j=1}^{N} \sum_{n=\pm 1}^{\pm\infty} \Gamma_i \Gamma_j \ln|z_i - z_j q^n|$$
$$+ \frac{1}{4\pi} \sum_{i,j=1}^{N} \sum_{n=-\infty}^{\infty} \Gamma_i \Gamma_j \ln\left|z_i \bar{z}_j - r_1^2 q^n\right|. \tag{122}$$

The first sum describes a direct interaction between N vortices, while the second sum corresponds to the interaction between vortices and their images of the even order, having the same sign of circulation. And the last sum is an interaction of vortices with the odd images, having opposite sign for the circulation.

9.2.1. The q-logarithmic form of the Hamiltonian. Despite of the clear vortex image picture of the above Hamiltonian, it contains an infinite sum. To avoid the divergency problem we can use the q-logarithmic form of equations of motion (117) as follows

$$H = -\frac{1}{4\pi} \sum_{i,j=1(i\neq j)}^{N} \Gamma_i \Gamma_j \ln|z_i - z_j| - \frac{1}{4\pi} \sum_{i,j=1}^{N} \Gamma_i \Gamma_j \ln \left| \frac{E_q\left(\frac{z_i}{(1-q)z_j}\right) E_q\left(\frac{z_j}{(1-q)z_i}\right)}{E_q\left(\frac{z_i \bar{z}_j}{(1-q)r_1^2}\right) E_q\left(\frac{r_2^2}{(1-q)z_i \bar{z}_j}\right)} \right|. \tag{123}$$

In this form the first sum describes the direct interaction between N vortices, while the second sum corresponds to the interaction of vortices with their images.

9.2.2. The Kirchhoff–Routh function. In the Hamiltonian (123) we collect to the separated sum all terms describing a vortex interaction with its own images, then we get the Hamiltonian function in the form of the Kirchhoff–Routh function [31, 32],

$$H = H_0 - \frac{1}{2\pi}\sum_{i<j}\Gamma_i\Gamma_j\left[\ln|z_i - z_j| + \ln\left|\frac{E_q\left(\frac{z_i}{(1-q)z_j}\right)E_q\left(\frac{z_j}{(1-q)z_i}\right)}{E_q\left(\frac{z_i\bar{z}_j}{(1-q)r_1^2}\right)E_q\left(\frac{r_2^2}{(1-q)z_i\bar{z}_j}\right)}\right|\right]$$
$$+\frac{1}{4\pi}\sum_{j=1}^{N}\Gamma_j\ln\left[E_q\left(\frac{|z_j|^2}{(1-q)r_1^2}\right)E_q\left(\frac{r_2^2}{(1-q)|z_j|^2}\right)\right], \tag{124}$$

where the constant

$$H_0 = -\frac{1}{2\pi}\sum_{j=1}^{N}\Gamma_j^2\ln E_q\left(\frac{1}{1-q}\right) = -\frac{1}{2\pi}\sum_{j=1}^{N}\Gamma_j^2\ln G_{\tilde{q}} \tag{125}$$

is given in terms of the Euler function (64) due to the relation

$$E_q\left(\frac{1}{1-q}\right) = \prod_{n=1}^{\infty}\left(1-\frac{1}{q^n}\right) = \prod_{n=1}^{\infty}\left(1-\frac{1}{(\sqrt{q})^{2n}}\right)\prod_{n=1}^{\infty}(1-\tilde{q}^{2n}) = G_{\tilde{q}}. \tag{126}$$

9.3. Green's function and vortex images

The hydrodynamic Green function [31], or Green's function of the first kind [29] is defined as

$$G_I = G + G_H^{(k)}, \tag{127}$$

where

$$G(z; z_k) = -\frac{1}{2\pi}\ln|z - z_k| \tag{128}$$

is the Green function in the unbounded plane and $G_H^{(k)}$ is a harmonic function

$$\Delta G_H^{(k)} = 0. \tag{129}$$

By choosing $G_H^{(k)} = -G$ at the boundary, one has $G_I = 0$. The relation between the stream function ψ and G_I [31],

$$\psi = \sum_{k=1}^{N}\Gamma_k G_I(z, z_k) \tag{130}$$

for the stream function (51), gives the Green function explicitly as

$$G_I(z, z_k) = -\frac{1}{2\pi}\sum_{n=-\infty}^{\infty}\ln\left|\frac{z - z_k q^n}{z - \frac{r_1^2}{\bar{z}_k}q^n}\right|. \tag{131}$$

In this sum, the term with $n = 0$ corresponds to the single vortex Green's function G (128), while other terms reflect the influence of boundaries

$$G_H^{(k)}(z; z_k) = -\frac{1}{2\pi}\sum_{n=\pm 1}^{\pm\infty}\ln|z - z_k q^n| + \frac{1}{2\pi}\sum_{n=-\infty}^{\infty}\ln\left|z - \frac{r_1^2}{\bar{z}_k}q^n\right|. \tag{132}$$

Here the even vortex images contribute to the first sum, while the odd images to the second one. The Green function G_I has several properties [31]. To prove them we can use q-exponential representation of the Green function.

9.4. Green's function and the q-exponential function

From (59) we find the stream function

$$\psi = -\frac{1}{2\pi}\sum_{k=1}^{N}\Gamma_k\left[\ln|z-z_k| + \ln\left|\frac{E_q\left(\frac{z}{(1-q)z_k}\right)E_q\left(\frac{z_k}{(1-q)z}\right)}{E_q\left(\frac{z\bar{z}_k}{(1-q)r_1^2}\right)E_q\left(\frac{r_2^2}{(1-q)z\bar{z}_k}\right)}\right|\right], \tag{133}$$

and from (130) the Green function

$$G_I(z,z_k) = -\frac{1}{2\pi}\ln|z-z_k| - \frac{1}{2\pi}\ln\left|\frac{E_q\left(\frac{z}{(1-q)z_k}\right)E_q\left(\frac{z_k}{(1-q)z}\right)}{r_2 E_q\left(\frac{z\bar{z}_k}{(1-q)r_1^2}\right)E_q\left(\frac{r_2^2}{(1-q)z\bar{z}_k}\right)}\right|, \tag{134}$$

where, to have suitable boundary values, we added the constant term $-(1/2\pi)\ln r_2$.

9.4.1. Symmetry of Green's function. Obviously, the Green function written in form (134) is symmetrical

$$G_I(z,z_k) = G_I(z_k,z). \tag{135}$$

9.4.2. Boundary values. At the outer circle $C_2 : |z| = r_2$, the Green function vanishes

$$G_I(z,z_k)|_{C_2} = 0 \tag{136}$$

and at the inner circle $C_1 : |z| = r_1$,

$$G_I(z,z_k)|_{C_1} = \frac{1}{2\pi}\ln\left|\frac{r_2}{z_k}\right|. \tag{137}$$

Differentiating (130) and using (47), we have the relation

$$\frac{\partial G_I}{\partial z} = \frac{1}{2\mathrm{i}}\bar{V}_k(z), \tag{138}$$

where $\bar{V}_k(z)$ is the complex velocity (39) due to the vortex at $z_k : \bar{V}(z) = \sum_{k=1}^{N}\bar{V}_k(z)$. We apply this relation to the normal derivative (like in (6)) then for the contour integral

$$\oint_{C_1}\frac{\partial G_I}{\partial n}\,\mathrm{d}s = -\oint_{C_1}\bar{V}_k(z) = -\sum_{l=1}^{\infty}\mathrm{Res}\left\{\bar{V}_k\left(z_k^{(l)}\right)\right\} \tag{139}$$

and by the residue theorem we have an infinite sum of $\pm\Gamma_k$ corresponding to all images of vortex z_k inside C_1 at positions $r_1^2/\bar{z}_k, z_k/q, r_1^2/\bar{z}_k q, z_k/q^2, \ldots,$ which compensate each other.

10. *N*-vortex polygons in the annular domain

In section 7 we studied a single vortex motion in the annular domain. Now we generalize this result by constructing a new exact configuration of *N* vortices. This problem is an extension of the Lord Kelvin's problem of stationary rotation of the system of *N* equal vortices located at the vertices of a regular polygon in the plane. For the cylindrical domain it was considered by Havelock [33]. And now we are studying this system in the annular domain. Recently, this problem has become relevant again, in connection with experiments on magnetized electron columns in plasma confined in a cylindrical trap [24, 25], and a mathematically similar problem for the point vortices in liquid helium in a rotating cylinder. Both of the problems could be studied in the annular domain. Moreover, an addition of one more cylinder to the origin of the cylindrical domain could modify the phases of particles by the Aharonov–Bohm effect [11].

10.1. Angular velocity

As is well known the system of N vortices in the plane, in addition to the energy, admits three integrals of motion, corresponding to the pair of translations and one rotation [28, 29]. However, for our problem in the annular domain only the energy and angular momentum, corresponding to rotation of the system, survives. Multiplying equation (116) by $\Gamma_k z_k$, then adding all equations together and with the complex conjugate ones, we find (see appendix A)

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_{k=1}^{N}\Gamma_k z_k \bar{z}_k\right) = 0, \tag{140}$$

which implies the conservation of angular momentum

$$I = \sum_{k=1}^{N}\Gamma_k z_k \bar{z}_k. \tag{141}$$

10.2. N-polygon solution

The conservation of angular momentum implies the existence of dynamical configuration of vortices, rotating with a fixed angular velocity. In the present paper, we restrict consideration to the simplest case when all circulations of vortices are equal to each other $\Gamma_k = \Gamma, k = 1, \ldots, N$. We suppose that all vortices are located at the same distance r from the center ($r_1 < r < r_2$), at vertices of the regular N polygon, described by roots of the unity. Then we have the ansatz

$$z_k(t) = r\,\mathrm{e}^{\mathrm{i}\omega t + \mathrm{i}\frac{2\pi}{N}k} \quad (k = 1, \ldots, N). \tag{142}$$

Substituting (142) into (117), we find the rotation frequency (for details, see appendix B)

$$\omega = \frac{\Gamma}{2\pi r^2}\left\{\frac{N-1}{2} + \frac{1}{q-1}\sum_{j=1}^{N}\left[\mathrm{Ln}_q\left(1 - \frac{r_2^2}{r^2}\,\mathrm{e}^{\mathrm{i}\frac{2\pi}{N}j}\right) - \mathrm{Ln}_q\left(1 - \frac{r^2}{r_1^2}\,\mathrm{e}^{-\mathrm{i}\frac{2\pi}{N}j}\right)\right]\right\}. \tag{143}$$

At first glance this expression looks complex valued. However, since $\mathrm{e}^{\mathrm{i}\frac{2\pi}{N}j}$ ($j = 1, \ldots, N$) are roots of the algebraic equation $z^N = 1$ with real coefficients, then all complex roots appear as complex conjugate pairs. Therefore, both q-logarithmic terms and hence the frequency ω, are real valued.

For the single vortex case, when $N = 1$, frequency (143) reduces to the one obtained in (87). However, vanishing of the frequency of rotation at the geometric mean distance $r = \sqrt{r_1 r_2}$, is no longer true for the N-vortex problem. Indeed at this distance the last two terms in (143) cancel each other and we obtain

$$\omega = \frac{\Gamma(N-1)}{4\pi r_1 r_2}. \tag{144}$$

It is worth noting that at this distance the vortex polygon is rotating with the frequency, independent of the parameter q, and hence independent of the ratio r_2/r_1. The cylinders r_1 and r_2 are images of each other in the cylinder $r = \sqrt{r_1 r_2}$. Moreover, the rotation frequency is the same for any pair of coaxial cylinders determining the point (r_1, r_2) on the hyperbola $r_2 = r^2/r_1$ with $r_2 > r_1$.

Existence of stationary polygon configuration with $\omega = 0$ is related to the existence of a solution of the following q-logarithmic equation:

$$\sum_{j=1}^{N}\left[\mathrm{Ln}_q\left(1 - \frac{r_2^2}{r^2}\,\mathrm{e}^{\mathrm{i}\frac{2\pi}{N}j}\right) - \mathrm{Ln}_q\left(1 - \frac{r^2}{r_1^2}\,\mathrm{e}^{-\mathrm{i}\frac{2\pi}{N}j}\right)\right] = \frac{N-1}{2}(q-1). \tag{145}$$

A particular solution of this equation when $N = 1$ and $r = \sqrt{r_1 r_2}$ is given by (89), (90). Existence of the solution for arbitrary N is an open problem. However, as we show in the following section, for sufficient large q the solution exists for any N.

10.3. The N-vortex polygon in the $q \to \infty$ limiting cases

The problem of the N-vortex polygon inside and outside of a cylinder have been considered by Havelock [33]. In this section, we show that his results appear from our new solution in the limit $q \to \infty$.

10.3.1. The N-vortex polygon outside of a cylinder. For $r_1 = \text{const}$ and $r_2^2 = q r_1^2 \to \infty$ the problem reduces to the N-vortex polygon outside the cylinder with radius r_1. In this limit, we have (for details of calculations, see appendix C)

$$\omega = \frac{\Gamma}{4\pi r^2}\left\{(3N-1) - \frac{2N}{1 - r_1^{2N}/r^{2N}}\right\} \tag{146}$$

which match exactly with result [28, 33].

10.3.2. The N-vortex polygon inside of a cylinder. For $r_2 = \text{const}$, the limit $q \to \infty$ implies $r_1^2 = r_2^2/q \to 0$. We have the following (see appendix C):

$$\omega = \frac{\Gamma}{4\pi r^2}\left\{(N-1) - \frac{2N}{1 - r_2^{2N}/r^{2N}}\right\}. \tag{147}$$

This coincides with [28, 33].

11. Conclusions

We have shown that the infinite set of images for a vortex in the annular domain between two coaxial cylinders is given completely in terms of the q-logarithmic function for complex velocity and in terms of the Jackson q-exponential function for the complex potential. Recent results on Padé approximation for the q-elementary functions [17] could be an efficient approximation in the vortex image problem, by restricting the number of images. For example in paper [19], in order to compute the q-logarithm, with reference to Sebah, the following formula is proposed:

$$\ln_q(1+z) = z\sum_{n=1}^{N}\frac{1}{q^n + z} + r_N(z), \tag{148}$$

where

$$r_N(z) = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}z^n}{q^{Nn}(q^n - 1)}, \tag{149}$$

with N being any positive integer. Then the application of this formula to our problem gives N-vortex images approximation. An advantage of this representation is in its very fast convergency which could be useful for numerical and analytical computations.

By conformal mapping of the annular region to the rectangular one and composing q-exponents, the stream function has been represented in terms of the Jacobi theta function and coincides exactly with known results. Since the annular region can be conformally mapped to the exterior of two cylinders in the plane, our solution also provides the solution of the latter problem in terms of q-elementary functions.

The q-elementary function representation of the problem allowed us to efficiently deal with an infinite set of vortex images. By the image picture, we constructed the Green function in the domain and the Hamiltonian structure in terms of these functions. As an application, we constructed the novel N-vortex polygon solution in the annular domain.

We would like to note that the elementary relation between the hydrodynamical problem and q-calculus considered in this paper could be applied to several physical situations with the same geometry, such as the electrostatic problem or the problem of anyons [11]. Instead of vortices we can consider problems with sources as well. Moreover, our image picture also suggests a q-lattice structure for the spherical shell, consisting of two spheres with radii r_1 and r_2 in n dimensions.

Finally, our results suggest an intriguing link between the point vortex problem and other areas of applied mathematics with which the subject *a priori* would seem to have no connection whatsoever. The notion of q-elementary functions has proven extremely powerful in other areas of mathematics, for example in combinatorics and theory of quantum groups, as well as in theoretical physics, for example in quantum field theory of many bodies. We think that interplay between model problems in fluid mechanics and basic mathematical structures, such as q-elementary functions will allow non-trivial results to be obtained concerning the vortex problem in the multiply connected domain.

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Appendix A. Conservation of angular momentum

Here we show the conservation of angular momentum (140). We rewrite system (116) in the form

$$\dot{\bar{z}}_k = \frac{1}{2\pi\mathrm{i}}\sum_{j=1}^{N}\frac{\Gamma_j(1-\delta_{jk})}{z_k - z_j} + \frac{1}{2\pi\mathrm{i}}\sum_{j=1}^{N}\sum_{n=\pm1}^{\pm\infty}\frac{\Gamma_j}{z_k - z_j q^n} - \frac{1}{2\pi\mathrm{i}}\sum_{j=1}^{N}\sum_{n=-\infty}^{\infty}\frac{\Gamma_j}{z_k - \frac{r_1^2}{\bar{z}_k}q^n} \tag{A.1}$$

$(k = 1, \ldots, N)$, where δ_{jk} is the Kroneker delta. Multiplying with $\Gamma_k z_k$ and adding together with its complex conjugate counterpart, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_{k=1}^{N}\Gamma_k z_k \bar{z}_k\right) = \frac{1}{2\pi\mathrm{i}}(A + B + C). \tag{A.2}$$

For the first term, we have the double sum

$$A = \sum_{k=1}^{N}\sum_{j=1}^{N}\Gamma_k\Gamma_j(1-\delta_{kj})\left(\frac{z_k}{z_k - z_j} - \frac{\bar{z}_k}{\bar{z}_k - \bar{z}_j}\right) \tag{A.3}$$

$$= \sum_{k=1}^{N}\sum_{j=1}^{N}\left(\frac{\Gamma_k\Gamma_j(1-\delta_{kj})}{|z_k - z_j|^2}\right)(\bar{z}_k z_j - \bar{z}_j z_k) = 0 \tag{A.4}$$

vanishing as a product of the symmetric and the skew-symmetric tensors. For the second term, we have

$$B = \sum_{k=1}^{N}\sum_{j=1}^{N}\sum_{n=\pm 1}^{\pm\infty}\Gamma_k\Gamma_j\left(\frac{z_k}{z_k - z_j q^n} - \frac{\bar{z}_k}{\bar{z}_k - \bar{z}_j q^n}\right) \tag{A.5}$$

$$= \sum_{n=1}^{\infty} q^n \sum_{k=1}^{N}\sum_{j=1}^{N}\left(\Gamma_k\Gamma_j\left[\frac{1}{|z_k - z_j q^n|^2} + \frac{1}{|z_j - z_k q^n|^2}\right]\right)(\bar{z}_k z_j - \bar{z}_j z_k) = 0 \tag{A.6}$$

for the same reason. In a similar way, for the last sum we have

$$C = \sum_{k=1}^{N}\sum_{j=1}^{N}\sum_{n=-\infty}^{\infty}\Gamma_k\Gamma_j\left(\frac{z_k\bar{z}_j}{z_k\bar{z}_j - r_1^2 q^n} - \frac{\bar{z}_k z_j}{\bar{z}_k z_j - r_1^2 q^n}\right) \tag{A.7}$$

$$= \sum_{n=-\infty}^{\infty} r_1^2 q^n \sum_{k=1}^{N}\sum_{j=1}^{N}\left(\frac{\Gamma_k\Gamma_j}{\left(z_k\bar{z}_j - r_1^2 q^n\right)\left(z_j\bar{z}_k - r_1^2 q^n\right)}\right)(\bar{z}_k z_j - \bar{z}_j z_k) = 0. \tag{A.8}$$

As a result

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_{k=1}^{N}\Gamma_k z_k \bar{z}_k\right) = 0. \tag{A.9}$$

Appendix B. *N*-vortex polygon frequency

Here we derive the frequency of rotation for the N-vortex polygon (143). We substitute

$$z_k(t) = r\,\mathrm{e}^{\mathrm{i}\omega t + \mathrm{i}\frac{2\pi}{N}k}(k = 1, \ldots, N) \tag{B.1}$$

to the system

$$\dot{z}_k = \frac{1}{2\pi\mathrm{i}}\sum_{j=1(j\neq k)}^{N}\frac{\Gamma_j}{z_k - z_j} + \frac{1}{2\pi\mathrm{i}}\sum_{j=1}^{N}\frac{\Gamma_j}{z_k(q-1)}\left[\mathrm{Ln}_q\left(1 - \frac{z_k}{z_j}\right) - \mathrm{Ln}_q\left(1 - \frac{z_j}{z_k}\right)\right]$$

$$- \frac{1}{2\pi\mathrm{i}}\sum_{j=1}^{N}\frac{\Gamma_j}{z_k(q-1)}\left[\mathrm{Ln}_q\left(1 - \frac{z_k\bar{z}_j}{r_1^2}\right) - \mathrm{Ln}_q\left(1 - \frac{r_2^2}{z_k\bar{z}_j}\right)\right](k = 1, \ldots, N). \tag{B.2}$$

Then for frequency ω, we have

$$\omega = \frac{\Gamma}{2\pi r^2}(A_1 + B_1 + C_1). \tag{B.3}$$

The first sum is standard and has appeared before for the N-vortex problem in the plane [30]. We have

$$A_1 = \sum_{j=1(j\neq k)}^{N}\frac{1}{1 - \mathrm{e}^{2\pi\mathrm{i}(j-k)/N}} = \sum_{j=1(j\neq k)}^{N}\frac{\left[1 - \cos\frac{2\pi(j-k)}{N}\right] - \mathrm{i}\sin\frac{2\pi(j-k)}{N}}{2\left(1 - \cos\frac{2\pi(j-k)}{N}\right)} \tag{B.4}$$

$$= \frac{1}{2}\sum_{j=1(j\neq k)}^{N}\left(1 - \mathrm{i}\cot\frac{\pi(j-k)}{N}\right). \tag{B.5}$$

Due to the relations

$$\sum_{j=1(j\neq k)}^{N} \cot\frac{\pi(j-k)}{N} = 0 \qquad (k=1,\ldots,N) \tag{B.6}$$

it becomes

$$A_1 = \frac{N-1}{2}. \tag{B.7}$$

Another way to get the same result is just to note that in the sum (B.4) the roots of unity are symmetric relative to k, so that every pair of complex conjugate roots gives the contribution

$$\frac{1}{1-\mathrm{e}^{2\pi \mathrm{i}l/N}} + \frac{1}{1-\mathrm{e}^{-2\pi \mathrm{i}l/N}} = 1, \tag{B.8}$$

while the real root (when N is even) at angle π gives $1/2$. The elegant proof of (B.7) can be done in terms of q-numbers. We define q-number $[N]_z$ with complex base z so that

$$[N]_z = \frac{z^N-1}{z-1} = 1+z+z^2+\cdots+z^{N-1}. \tag{B.9}$$

As a complex polynomial this number has $N-1$ primitive roots of unity $z_l = \exp(2\pi\mathrm{i}l/N)$, $l=1,\ldots,N-1$,

$$[N]_z = \frac{z^N-1}{z-1} = (z-z_1)(z-z_2)\cdots(z-z_{N-1}). \tag{B.10}$$

The logarithmic derivative of this relation gives

$$\frac{\frac{\mathrm{d}}{\mathrm{d}z}[N]_z}{[N]_z} = \sum_{l=1}^{N-1}\frac{1}{z-z_l}. \tag{B.11}$$

Taking the limit $z\to 1$ from both sides, we find

$$\sum_{l=1}^{N-1}\frac{1}{1-\mathrm{e}^{2\pi\mathrm{i}l/N}} = \frac{N-1}{2}. \tag{B.12}$$

To calculate the second term in (B.3) we need the following lemmas.

Lemma 1. *The sum*

$$\sum_{j=1}^{N}\mathrm{Ln}_q\left(1-\mathrm{e}^{\pm\mathrm{i}\frac{2\pi}{N}(k-j)}\right) \qquad (k=1,\ldots,N) \tag{B.13}$$

is real. It follows easily from the fact that complex roots of unity appear by complex conjugate pairs.

Lemma 2. *The sum*

$$\sum_{j=1}^{N}\mathrm{Ln}_q\left(1-\mathrm{e}^{\pm\mathrm{i}\frac{2\pi}{N}(k-j)}\right) = N\,\mathrm{Ln}_q 0 = -NH(q) \qquad (k=1,\ldots,N), \tag{B.14}$$

where $H(q)$ is the q-harmonic series (83), converging for $q>1$.

Proof. Expanding, we have

$$\sum_{j=1}^{N}\mathrm{Ln}_q\left(1-\mathrm{e}^{\pm\mathrm{i}\frac{2\pi}{N}(k-j)}\right) = -\sum_{j=1}^{N}\sum_{n=1}^{\infty}\frac{1}{[n]}\mathrm{e}^{\pm\mathrm{i}\frac{2\pi}{N}(k-j)n} = -\sum_{n=1}^{\infty}\frac{1}{[n]}\mathrm{e}^{\pm\mathrm{i}\frac{2\pi}{N}kn}\sum_{j=1}^{N}\left(\mathrm{e}^{\pm\mathrm{i}\frac{2\pi}{N}n}\right)^j. \tag{B.15}$$

To estimate the last sum

$$S \equiv \sum_{j=1}^{N} \left(e^{\pm i \frac{2\pi}{N} n}\right)^j \tag{B.16}$$

we use the identity

$$z(1+z+z^2+\cdots+z^{N-1}) = z\frac{z^N-1}{z-1} = z(z-z_1)(z-z_2)\cdots(z-z_{N-1}). \tag{B.17}$$

We have two options. If (a) n is not divisible by N, $n \neq mN$ (for m-integer), then $e^{\pm i \frac{2\pi}{N} n}$ is the primitive root of unity. In this case from the above identity follows $S = 0$. (b) n is divisible by N, $n = mN$ (m is an integer), then $e^{\pm i \frac{2\pi}{N} mN} = e^{\pm i 2\pi m} = 1$ and as a result $S = N$. Substituting to (B.15) we arrive with $-N\sum_{m=1}^{\infty} 1/[m] = -NH(q)$.

Due to lemma 2, we find that in (B.3)

$$B_1 = \frac{1}{q-1}\left[\sum_{j=1}^{N} \mathrm{Ln}_q\left(1 - e^{i\frac{2\pi}{N}(k-j)}\right) - \sum_{j=1}^{N} \mathrm{Ln}_q\left(1 - e^{-i\frac{2\pi}{N}(k-j)}\right)\right] = 0. \tag{B.18}$$

It is easy to see that the last term in (B.3)

$$C_1 = \frac{1}{q-1}\left[\sum_{j=1}^{N} \mathrm{Ln}_q\left(1 - \frac{r_2^2}{r^2} e^{-i\frac{2\pi}{N}(k-j)}\right) - \sum_{j=1}^{N} \mathrm{Ln}_q\left(1 - \frac{r^2}{r_1^2} e^{i\frac{2\pi}{N}(k-j)}\right)\right] \tag{B.19}$$

is independent of k, so that we have

$$C_1 = \frac{1}{q-1}\sum_{j=1}^{N}\left[\mathrm{Ln}_q\left(1 - \frac{r_2^2}{r^2} e^{i\frac{2\pi}{N}j}\right) - \mathrm{Ln}_q\left(1 - \frac{r^2}{r_1^2} e^{-i\frac{2\pi}{N}j}\right)\right]. \tag{B.20}$$

Substituting into (B.3), finally we find frequency (143). □

Appendix C. $q \to \infty$ limit for N-vortex polygon

(1) When $r_1 = \text{const}$, $r_2^2 = qr_1^2 \to \infty$ to estimate q-logarithms in (143)

$$\omega = \frac{\Gamma}{2\pi r^2}\left\{\frac{N-1}{2} + \frac{1}{q-1}\sum_{j=1}^{N}\left[\mathrm{Ln}_q\left(1 - q\frac{r_1^2}{r^2} e^{i\frac{2\pi}{N}j}\right) - \mathrm{Ln}_q\left(1 - \frac{r^2}{r_1^2} e^{-i\frac{2\pi}{N}j}\right)\right]\right\}, \tag{C.1}$$

we use (97) and (99) so that

$$\omega = \frac{\Gamma}{2\pi r^2}\left\{\frac{N-1}{2} + \sum_{j=1}^{N}\frac{r_1^2 e^{i\frac{2\pi}{N}j}}{r_1^2 e^{i\frac{2\pi}{N}j} - r^2}\right\} = \frac{\Gamma}{4\pi r^2}\left\{3N - 1 + 2r^2\sum_{j=1}^{N}\frac{1}{r_1^2 e^{i\frac{2\pi}{N}j} - r^2}\right\}. \tag{C.2}$$

To estimate the sum, we consider the logarithmic derivative

$$\frac{P'(z)}{P(z)} = \sum_{j=1}^{N}\frac{1}{z - z_j} \tag{C.3}$$

for polynomial $P(z) = z^N - r_1^{2N}/r^{2N}$. Taking the limit $z \to 1$ we have

$$\sum_{j=1}^{N}\frac{1}{1 - \frac{r_1^2}{r^2} e^{i2\pi j/N}} = \frac{N}{1 - \frac{r_1^{2N}}{r^{2N}}}. \tag{C.4}$$

As a result, we find frequency (144) coinciding with [3, 28]:

$$\omega = \frac{\Gamma}{4\pi r^2}\left\{(3N-1) - \frac{2N}{1 - r_1^{2N}/r^{2N}}\right\}. \tag{C.5}$$

(2) When $r_2 = \text{const}$, $r_1^2 = r_2^2/q \to 0$, the frequency

$$\omega = \frac{\Gamma}{2\pi r^2}\left\{\frac{N-1}{2} + \frac{1}{q-1}\sum_{j=1}^{N}\left[\mathrm{Ln}_q\left(1 - \frac{r_2^2}{r^2}\,\mathrm{e}^{\mathrm{i}\frac{2\pi}{N}j}\right) - \mathrm{Ln}_q\left(1 - q\frac{r^2}{r_2^2}\,\mathrm{e}^{-\mathrm{i}\frac{2\pi}{N}j}\right)\right]\right\} \tag{C.6}$$

by (97) and (99) transforms to

$$\omega = \frac{\Gamma}{2\pi r^2}\left\{\frac{N-1}{2} - \sum_{j=1}^{N}\frac{r^2\mathrm{e}^{-\mathrm{i}\frac{2\pi}{N}j}}{r^2\mathrm{e}^{-\mathrm{i}\frac{2\pi}{N}j} - r_2^2}\right\}. \tag{C.7}$$

Estimating the sum in a similar way as above, we have

$$-\sum_{j=1}^{N}\frac{r^2\,\mathrm{e}^{-\mathrm{i}\frac{2\pi}{N}j}}{r^2\,\mathrm{e}^{-\mathrm{i}\frac{2\pi}{N}j} - r_2^2} = -\sum_{j=1}^{N}\frac{1}{1 - \frac{r_2^2}{r^2}\,\mathrm{e}^{\mathrm{i}2\pi j/N}} = -\frac{N}{1 - \frac{r_2^{2N}}{r^{2N}}}. \tag{C.8}$$

Finally, we get frequency (147):

$$\omega = \frac{\Gamma}{4\pi r^2}\left\{(N-1) - \frac{2N}{1 - r_2^{2N}/r^{2N}}\right\}. \tag{C.9}$$

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